

Partial inverse problems for Sturm-Liouville operators on trees

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Abstract. In this paper, inverse spectral problems for Sturm-Liouville operators on a tree (a graph without cycles) are studied. We show that if the potential on an edge is known a priori, then $b - 1$ spectral sets uniquely determine the potential functions on a tree with b external edges. Constructive solutions, based on the method of spectral mappings, are provided for the considered inverse problems.

Keywords: quantum graphs; Sturm-Liouville operators; inverse spectral problems; method of spectral mappings.

AMS Mathematics Subject Classification (2010): 34A55 47E05 34B45 34L40

1. Introduction.

This paper concerns the theory of inverse spectral problems for Sturm-Liouville operators on geometrical graphs. Inverse problems consist in recovering differential operators from their spectral characteristics. Differential operators on graphs (quantum graphs) have applications in various fields of science and engineering (mechanics, chemistry, electronics, nanoscale technology and others) and attract a considerable attention of mathematicians in recent years. There is an extensive literature devoted to differential operators on graphs and their applications, we mention only some research papers and surveys [1, 2, 3, 4, 5, 6, 16].

There are different kinds of inverse problems studied for quantum graphs, one of them is to recover the coefficients of the operator while some information is known a priori. This paper is focused on the reconstruction of the potential of the Sturm-Liouville operator on a tree (a graph without cycles) with a prescribed structure and standard matching conditions in the vertices. V.A. Yurko [7, 8] studied such inverse problems on trees by the Weyl vector, the system of spectra and the spectral data. These problems are generalizations of the well-studied inverse problems for Sturm-Liouville operators on a finite interval (see monographs [9, 10, 11, 12] and references therein). By the method of spectral mappings [12, 13], V.A. Yurko proved uniqueness theorems and developed a constructive algorithm for solution of inverse problems on trees.

In this paper, we formulate and solve partial inverse problems for the Sturm-Liouville operator on the tree. We suppose that the Sturm-Liouville potential is known on the part of the graph and show that we need less data to recover the potential on the remaining part. We know the only work [14] in this direction, where the potential is known on a half of one edge and completely on the other edges of the star-shaped graph, and the author solves the Hochstadt-Lieberman-type problem [15] by a part of the spectrum.

In this paper, we assume that the potential is known on one edge of a tree, then reconstruct the potential on the remaining part by the system of spectra or the Weyl functions. By developing the ideas of V.A. Yurko [7, 8], we show that one needs one less spectral set or one less Weyl function for the solution of the partial inverse problem. We consider separately the cases of boundary and internal edges, present constructive solutions and corresponding uniqueness theorems for both of them.

The results of this paper can be generalized to the case, when the potential is known on several edges. However, in this case the number of given spectra, sufficient to recover the potential on the whole graph, depends not only on the number of these edges, but also on their location (see the example in Section 5). We note that the method of spectral mappings works also for graphs with cycles (see [16]), so one can generalize our results in this direction.

The paper is organized as follows. In *Section 2*, we introduce the notation and briefly describe the solution of inverse problems on trees by V.A. Yurko [7, 8]. In *Section 3*, we formulate our main results and outline their constructive solutions. *Section 4* contains proofs of the technical lemmas from Section 3. In *Section 5* we illustrate our method by an example.

2. Inverse problems on a tree

In this section, we introduce the notation and provide the main results of V.A. Yurko on the inverse problems on trees (see works [7, 8] for more details).

Consider a compact tree G with the vertices $V = \{v_i\}_{i=1}^{m+1}$ and edges $E = \{e_j\}_{j=1}^m$. For each vertex $v \in V$, we denote the set of edges associated with v by E_v and call the size of E_v the *degree* of v . Assume that the tree G does not contain vertices of degree 2. The vertices of degree 1 are called *boundary vertices*. Denote the set of boundary vertices of the graph G by ∂G . For the sake of convenience, let each boundary vertex v_i be an end of the edge e_i , such edges are called *boundary edges*. All other vertices and edges are called *internal*. Let the vertex $v_r \in \partial G$ be *the root* of the tree.

Each edge $e_j \in E$ is viewed as a segment $[0, T_j]$ and is parametrized by the parameter $x_j \in [0, T_j]$. The value $x_j = 0$ correspond to one of the end vertices of the edge e_j , and $x_j = T_j$ corresponds to another one. For a boundary edge, the end $x_j = 0$ corresponds to the boundary vertex v_j .

A function on the tree G can be represented as a vector function $y = [y_j]_{j=1}^m$, where $y_j = y_j(x_j)$, $x_j \in [0, T_j]$, $j = \overline{1, m}$. Let $e_j = [v_i, v_k]$, i.e. the vertex v_i corresponds to the end $x_j = 0$ and the vertex v_k corresponds to $x_j = T_j$. Introduce the following notation

$$\begin{aligned} y_j(v_i) &= y_j(0), & y_j(v_k) &= y_j(T_j), \\ y'_j(v_i) &= y'_j(0), & y'_j(v_k) &= -y'_j(T_j). \end{aligned}$$

If $v_i \in \partial G$, we omit the index of the edge and write $y(v_i)$ and $y'(v_i)$.

Consider the Sturm-Liouville equation on G :

$$-y''_j + q_j(x_j)y_j = \lambda y_j, \quad x_j \in [0, T_j], \quad j = \overline{1, m}. \quad (1)$$

where λ is the spectral parameter, $q_j \in L[0, T_j]$. We call the function $q = [q_j]_{j=1}^m$ the *potential* on the graph G . The functions y_j, y'_j are absolutely continuous on the segments $[0, T_j]$ and satisfy the *standard matching conditions* in the internal vertices $v \in V \setminus \partial G$:

$$\begin{cases} y_j(v) = y_k(v), & e_j, e_k \in E_v \quad (\text{continuity condition}), \\ \sum_{e_j \in E_v} y'_j(v) = 0, & (\text{Kirchhoff's condition}). \end{cases} \quad (2)$$

Let L_0 and $L_k, v_k \in \partial G$, be the boundary value problem for system (1) with the matching conditions (2) and the following conditions in the boundary vertices:

$$L_0: y(v_i) = 0, \quad v_i \in \partial G, \quad (3)$$

$$L_k: y'(v_k) = 0, \quad y(v_i) = 0, \quad v_i \in \partial G \setminus \{v_k\}. \quad (4)$$

It is well-known, that the problems L_k have discrete spectra, which are the countable sets of eigenvalues $\Lambda_k = \{\lambda_{ks}\}_{s=1}^\infty$, $k = 0$ or $v_k \in \partial G$.

Fix a boundary vertex $v_k \in \partial G$. Let $\Psi_k = [\psi_{kj}]_{j=1}^m$, $\psi_{kj} = \psi_{kj}(x_j, \lambda)$, be the solution of the system (1), satisfying the matching conditions (2) and the boundary conditions

$$\psi_{kk}(0, \lambda) = 1, \quad \psi_{kj}(0, \lambda) = 0, \quad v_j \in \partial G \setminus \{v_k\}.$$

Denote $M_k(\lambda) = \psi'_{kk}(0, \lambda)$. The functions Ψ_k and M_k are called *the Weyl solution* and *the Weyl function* of (1) with respect to the boundary vertex v_k , respectively. The notion of the Weyl

function for the tree generalizes the notion of the Weyl function (m -function) for the classical Sturm-Liouville operator on a finite interval [9, 12]. If the tree G consists of only one edge, then $M_k(\lambda)$ coincide with the classical Weyl function.

Consider the following inverse problems.

Inverse Problem 1. *Given the spectra $\Lambda_0, \Lambda_k, v_k \in \partial G \setminus \{v_r\}$, construct the potential q on the tree G .*

Inverse Problem 2. *Given the Weyl functions $M_k(\lambda), v_k \in \partial G \setminus \{v_r\}$, construct the potential q on the tree G .*

Note that if the number of boundary vertices is b , then one needs b spectra or $b - 1$ Weyl functions to recover the potential. We do not require the data associated with the root v_r .

There is a close relation between Inverse Problems 1 and 2. The Weyl functions can be represented in the form

$$M_k(\lambda) = -\frac{\Delta_k(\lambda)}{\Delta_0(\lambda)}, \quad v_k \in \partial G, \quad (5)$$

where $\Delta_k(\lambda)$ are characteristic functions of the boundary value problems L_k . If the eigenvalues Λ_k are known, one can construct characteristic functions as infinite products by Hadamard's theorem. Thus, with the system of spectra, one can obtain the Weyl functions and reduce Inverse Problem 1 to Inverse Problem 2.

V.A. Yurko has proved, that Inverse Problems 1 and 2 are uniquely solvable, and provided a constructive algorithm for the solution by the method of spectral mappings [12]. In the remaining of this section, we shall briefly describe his algorithm. Let the Weyl functions $M_k(\lambda), v_k \in \partial G \setminus \{v_r\}$ be given. Consider the following auxiliary problem.

Problem IP(k). Given $M_k(\lambda)$, construct the potential $q_k(x_k)$ on the edge e_k .

Note that this problem is not equivalent to the inverse problem on the finite interval, since the Weyl function $M_k(\lambda)$ contains information from the whole graph. However, it can be solved uniquely by the method of spectral mappings, and the potential on the boundary edges can be recovered. Then V.A. Yurko used so-called μ -procedure to recover the potential on the internal edges. We reformulate these ideas in the form, which is more convenient for us in the future.

Theorem 1. *Let v be an internal vertex, connected with the set of boundary vertices $V' \subset \partial G \setminus \{v_r\}$ and only one other vertex. Suppose the potentials q_k on the edges e_k are known for all $v_k \in V'$, as well as a Weyl function $M_k(\lambda)$ for at least one vertex from the set V' . Denote G' the graph by removing the vertices $v_k \in V'$ together with the corresponding edges e_k from the graph G . Then the Weyl function for the graph G' with respect the the vertex v can be determined from the given information.*

Applying Theorem 1, one can cut the boundary edges off, until the potential will be recovered on the whole graph.

3. Partial inverse problems

In this section, the main results of the paper are formulated. We assume that the potential is known on one edge of the tree and formulate partial inverse problems. We consider separately the cases of boundary and internal edge. The first one appears to be trivial, for the second one we describe the procedure of the constructive solution. For the convenience of the reader, the proofs of the technical lemmas are provided in Section 4.

Inverse Problem 3. *Let e_f be a boundary edge ($f \neq r$). Given the potential q_f on the edge e_f and the spectra $\Lambda_0, \Lambda_k, v_k \in \partial G \setminus \{v_f, v_r\}$. Construct the potential q on the tree G .*

The solution of Inverse Problem 3 is a slight modification of the method described in Section 2. From $\Lambda_0, \Lambda_k, v_k \in \partial G \setminus \{v_f, v_r\}$, we easily construct the potentials q_k for $v_k \in \partial G \setminus \{v_f, v_r\}$. The potential q_f is known, so we can apply Theorem 1 iteratively and recover the potential on G .

Now let e_f be an internal edge. If this edge is removed, the graph splits into two parts, call them P_1 and P_2 . Let ∂P_1 and ∂P_2 be the sets of boundary vertices of P_1 and P_2 , respectively. Fix two arbitrary vertices $v_{r1} \in \partial P_1$ and $v_{r2} \in \partial P_2$.

Inverse Problem 4. *Given the potential q_f on the internal edge e_f , the spectra $\Lambda_0, \Lambda_k, v_k \in \partial G \setminus \{v_{r1}, v_{r2}\}$. Construct the potential q on the tree G .*

Solution of Inverse Problem 4. For simplicity, we assume that the ends of the edge e_f have degree 3. The general case requires minor modifications. If one splits each of the ends of e_f into three vertices, the tree splits into five subtrees $G_i, i = \overline{1, 5}$, such that $v_{r1} \in G_2, v_{r2} \in G_5$, and G_3 contains the only edge e_f (see fig. 1). Let v_1 and v_4 are arbitrary boundary vertices of the trees G_1 and G_4 (different from the ends of e_f), $v_{r1} = v_2, v_{r2} = v_5, e_f = [v_3, v_6]$.

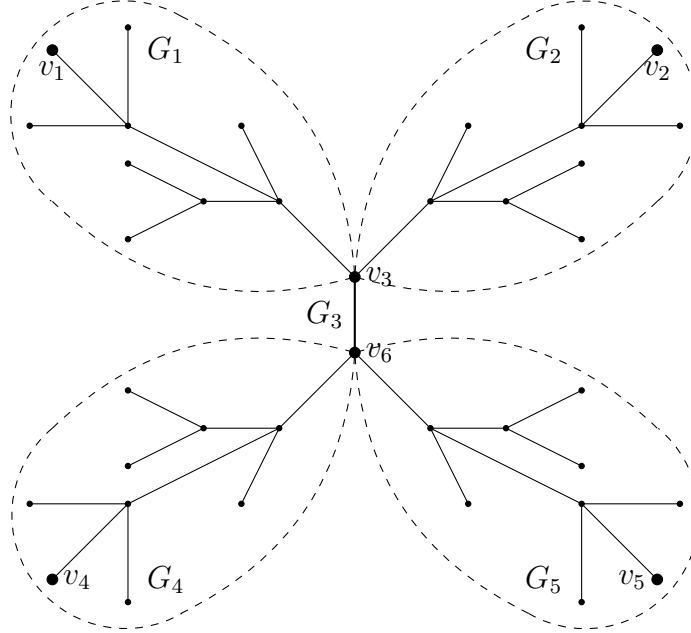


Figure 1

Step 1. Construct the characteristic functions $\Delta_k(\lambda)$ by the given spectra $\Lambda_k, k = 0$ and $v_k \in \partial G \setminus \{v_2, v_5\}$. Find $M_k(\lambda)$ by formula (5).

Step 2. Consider trees G_1 and G_4 . Recover the potential q on the edges of G_1 and G_4 , using the solutions of the problems IP(k) for $v_k \in \partial G_1 \setminus \{v_3\}$ and $v_k \in \partial G_4 \setminus \{v_6\}$, and then applying Theorem 1 iteratively.

Step 3. Introduce the characteristic functions of the boundary value problems for the Sturm-Liouville equations (1) on the graphs G_1 - G_5 with the standard matching conditions (2) in internal vertices and the following conditions in the boundary vertices:

$$\begin{aligned} \text{graph } G_1 & \begin{cases} \Delta_1^{DD}(\lambda): & y(v_k) = 0, & v_k \in \partial G_1, \\ \Delta_1^{ND}(\lambda): & y'(v_1) = 0, & y(v_k) = 0, & v_k \in \partial G_1 \setminus \{v_1\}, \\ \Delta_1^{DN}(\lambda): & y'(v_3) = 0, & y(v_k) = 0, & v_k \in \partial G_1 \setminus \{v_3\}, \\ \Delta_1^{NN}(\lambda): & y'(v_1) = 0, & y'(v_3) = 0, & y(v_k) = 0, & v_k \in \partial G_1 \setminus \{v_1, v_3\}. \end{cases} \\ \text{graph } G_2 & \begin{cases} \Delta_2^D(\lambda): & y(v_k) = 0, & v_k \in \partial G_2, \\ \Delta_2^N(\lambda): & y'(v_3) = 0, & y(v_k) = 0, & v_k \in \partial G_2 \setminus \{v_3\}. \end{cases} \end{aligned}$$

$$\begin{aligned}
\text{graph } G_3 & \begin{cases} \Delta_3^{DD}(\lambda): & y(v_3) = 0, & y(v_6) = 0, \\ \Delta_3^{ND}(\lambda): & y'(v_3) = 0, & y(v_6) = 0, \\ \Delta_3^{DN}(\lambda): & y(v_3) = 0, & y'(v_6) = 0, \\ \Delta_3^{NN}(\lambda): & y'(v_3) = 0, & y'(v_6) = 0. \end{cases} \\
\text{graph } G_4 & \begin{cases} \Delta_4^{DD}(\lambda): & y(v_k) = 0, & v_k \in \partial G_4, \\ \Delta_4^{ND}(\lambda): & y'(v_4) = 0, & y(v_k) = 0, & v_k \in \partial G_4 \setminus \{v_4\}, \\ \Delta_4^{DN}(\lambda): & y'(v_6) = 0, & y(v_k) = 0, & v_k \in \partial G_4 \setminus \{v_6\}, \\ \Delta_4^{NN}(\lambda): & y'(v_4) = 0, & y'(v_3) = 0, & y(v_k) = 0, & v_k \in \partial G_4 \setminus \{v_4, v_6\}. \end{cases} \\
\text{graph } G_5 & \begin{cases} \Delta_5^D(\lambda): & y(v_k) = 0, & v_k \in \partial G_5, \\ \Delta_5^N(\lambda): & y'(v_6) = 0, & y(v_k) = 0, & v_k \in \partial G_5 \setminus \{v_6\}. \end{cases}
\end{aligned}$$

Lemma 1. *The following relation holds*

$$\Delta_0(\lambda) = \begin{vmatrix} \Delta_1^{DD}(\lambda) & -\Delta_2^D(\lambda) & 0 & 0 & 0 & 0 \\ 0 & \Delta_2^D(\lambda) & -1 & 0 & 0 & 0 \\ \Delta_1^{DN}(\lambda) & \Delta_2^N(\lambda) & 0 & -1 & 0 & 0 \\ 0 & 0 & \Delta_3^{ND}(\lambda) & \Delta_3^{DD}(\lambda) & -\Delta_4^{DD}(\lambda) & 0 \\ 0 & 0 & 0 & 0 & \Delta_4^{DD}(\lambda) & -\Delta_5^D(\lambda) \\ 0 & 0 & \Delta_3^{NN}(\lambda) & \Delta_3^{DN}(\lambda) & \Delta_4^{DN}(\lambda) & \Delta_5^N(\lambda) \end{vmatrix}. \quad (6)$$

If one changes $\Delta_1^{DD}(\lambda)$ to $\Delta_1^{ND}(\lambda)$ and $\Delta_1^{DN}(\lambda)$ to $\Delta_1^{NN}(\lambda)$, he obtains the determinant equal to $\Delta_1(\lambda)$. Similarly, if one changes $\Delta_4^{DD}(\lambda)$ to $\Delta_4^{ND}(\lambda)$ and $\Delta_4^{DN}(\lambda)$ to $\Delta_4^{NN}(\lambda)$, he gets $\Delta_4(\lambda)$.

Step 4. Note that the functions $\Delta_0(\lambda)$, $\Delta_1(\lambda)$, $\Delta_4(\lambda)$ are known from Step 1. Since we know the potential on the graphs G_1 , G_4 (from Step 2) and G_3 (given a priori), we can easily construct the characteristic functions for these graphs. Consider the relation (6) and similar relations for $\Delta_1(\lambda)$ and $\Delta_4(\lambda)$ as a system of equations with respect to $\Delta_2^D(\lambda)$, $\Delta_2^N(\lambda)$, $\Delta_5^D(\lambda)$ and $\Delta_5^N(\lambda)$ in the following form

$$\begin{cases} a_{11}\Delta_2^D\Delta_5^D + a_{12}\Delta_2^N\Delta_5^D + a_{13}\Delta_2^D\Delta_5^N + a_{14}\Delta_2^N\Delta_5^N = \Delta_0, \\ a_{21}\Delta_2^D\Delta_5^D + a_{22}\Delta_2^N\Delta_5^D + a_{23}\Delta_2^D\Delta_5^N + a_{24}\Delta_2^N\Delta_5^N = \Delta_1, \\ a_{31}\Delta_2^D\Delta_5^D + a_{32}\Delta_2^N\Delta_5^D + a_{33}\Delta_2^D\Delta_5^N + a_{34}\Delta_2^N\Delta_5^N = \Delta_4, \end{cases} \quad (7)$$

where $a_{ij} = a_{ij}(\lambda)$, $i = \overline{1,3}$, $j = \overline{1,4}$, are known coefficients.

Step 5. Multiply the first equation of (7) by Δ_1 and subtract the second equations, multiplied by Δ_0 . Apply the similar transform to the first and the third equations. Then we obtain the system

$$\begin{cases} b_{11}\Delta_2^D\Delta_5^D + b_{12}\Delta_2^N\Delta_5^D + b_{13}\Delta_2^D\Delta_5^N + b_{14}\Delta_2^N\Delta_5^N = 0, \\ b_{21}\Delta_2^D\Delta_5^D + b_{22}\Delta_2^N\Delta_5^D + b_{23}\Delta_2^D\Delta_5^N + b_{24}\Delta_2^N\Delta_5^N = 0, \end{cases}$$

where

$$b_{1i} = a_{1i}\Delta_1 - a_{2i}\Delta_0, \quad b_{2i} = a_{1i}\Delta_4 - a_{3i}\Delta_0, \quad i = \overline{1,4}. \quad (8)$$

Divide both equations by $\Delta_2^D\Delta_5^D$.

$$b_{i1} + b_{i2}\tilde{M}_2 + b_{i3}\tilde{M}_5 + b_{i4}\tilde{M}_2\tilde{M}_5 = 0, \quad i = 1, 2, \quad (9)$$

where

$$\tilde{M}_2(\lambda) = \frac{\Delta_2^N(\lambda)}{\Delta_2^D(\lambda)}, \quad \tilde{M}_5(\lambda) = \frac{\Delta_5^N(\lambda)}{\Delta_5^D(\lambda)}$$

are (up to the sign) the Weyl functions for the subtrees G_2 and G_5 associated with the vertices v_3 and v_6 , respectively.

Step 6. From the system (9) we easily derive

$$\tilde{M}_5 = -\frac{b_{i1} + b_{i2}\tilde{M}_2}{b_{i3} + b_{i4}\tilde{M}_2}, \quad i = 1, 2.$$

Hence

$$(b_{11} + b_{12}\tilde{M}_2)(b_{23} + b_{24}\tilde{M}_2) = (b_{21} + b_{22}\tilde{M}_2)(b_{13} + b_{14}\tilde{M}_2).$$

Finally, we obtain the quadratic equation with respect to $\tilde{M}_2(\lambda)$:

$$A(\lambda)\tilde{M}_2^2(\lambda) + B(\lambda)\tilde{M}_2(\lambda) + C(\lambda) = 0, \quad (10)$$

with analytic coefficients $A(\lambda)$, $B(\lambda)$, $C(\lambda)$:

$$\begin{aligned} A &= b_{12}b_{24} - b_{22}b_{14}, \\ B &= b_{11}b_{24} + b_{12}b_{23} - b_{21}b_{14} - b_{22}b_{13}, \\ C &= b_{11}b_{23} - b_{21}b_{13}. \end{aligned} \quad (11)$$

Step 7. Consider the Sturm-Liouville equation (1) on the tree G with the potential $q = 0$. Implement Steps 1–6 for this case and obtain the quadratic equation

$$A_0(\lambda)\tilde{M}_{20}^2(\lambda) + B_0(\lambda)\tilde{M}_{20}(\lambda) + C_0(\lambda) = 0, \quad (12)$$

analogous to (10). Denote $\rho = \sqrt{\lambda}$, $\operatorname{Re} \rho \geq 0$, $S_\delta := \{\rho: \operatorname{Re} \rho \geq 0, |\operatorname{Im} \rho| \leq \delta\}$, $\delta > 0$, $[1] = 1 + O(\rho^{-1})$. Let $f(\rho^2)$ be an analytic function and $\varepsilon > 0$. Denote $Z_\varepsilon(f) := \{\rho: |f(\rho^2)| \geq \varepsilon\}$.

Lemma 2. *The following asymptotic relations hold*

$$A(\lambda) = A_0(\lambda)[1], \quad B(\lambda) = B_0(\lambda)[1], \quad C(\lambda) = C_0(\lambda)[1], \quad \rho \in S_\delta \cap Z_\varepsilon(A_0B_0C_0), \quad |\rho| \rightarrow \infty.$$

Consequently, $D(\lambda) = D_0(\lambda)[1]$ for $\rho \in S_\delta \cap Z_\varepsilon(D_0)$, $|\rho| \rightarrow \infty$, where $D(\lambda)$ and $D_0(\lambda)$ are discriminants of equations (10) and (12), respectively.

Lemma 3. $A_0(\lambda) \neq 0$, $D_0(\lambda) \neq 0$.

It follows from Lemmas 2 and 3, that the quadratic equation (10) does not degenerate for $\rho \in S_\delta \cap Z_\varepsilon(A_0D_0)$, and two roots of (10) are different by asymptotics as $|\rho| \rightarrow \infty$. One can easily find an asymptotic representation of $\tilde{M}_2(\lambda)$ for any particular graph and choose the correct root of (10) on some region of S_δ for sufficiently large $|\rho|$. Then the function $\tilde{M}_2(\lambda)$ can be constructed for all $\lambda \in \mathbb{C}$ except its singularities by analytic continuation. Similarly one can find $\tilde{M}_5(\lambda)$.

Step 8. Consider the tree G_2 with the root v_2 . Solve problems IP(k) by $M_k(\lambda)$, $v_k \in \partial G_2 \setminus \{v_2, v_6\}$, and by $\tilde{M}_2(\lambda)$ for v_3 , obtain the potential on the boundary edges except e_2 . Then apply the cutting of boundary edges by Theorem 1 and recover the potential q on G_2 . The subtree G_5 can be treated similarly.

Thus, we recovered the potential q on the whole graph G . In parallel, we have proved the following uniqueness theorem.

Theorem 2. *Let the potential q_f on the edge e_f ($f \neq r$) be known.*

(i) *If e_f is a boundary edge, the spectra Λ_0 , Λ_k , $v_k \in \partial G \setminus \{v_f, v_r\}$, uniquely determine the potential q on the whole graph G .*

(ii) *If e_f is an internal edge, the spectra Λ_0 , Λ_k , $v_k \in \partial G \setminus \{v_{r1}, v_{r2}\}$ uniquely determine the potential q on the whole graph G .*

Using the described methods with some technical modifications, one can solve partial inverse problems by Weyl functions.

Inverse Problem 5. Let e_f be a boundary edge ($f \neq r$). Given the potential q_f on the edge e_f and the Weyl functions $M_k(\lambda)$, $v_k \in \partial G \setminus \{v_f, v_r\}$. Construct the potential q on the tree G .

Inverse Problem 6. Given the potential q_f on the internal edge e_f , the Weyl functions $M_k(\lambda)$ $v_k \in \partial G \setminus \{v_{r1}, v_{r2}\}$. Construct the potential q on the tree G .

Thus, if the number of boundary edges is b and the potential is known on one edge (boundary or internal), $b - 2$ Weyl functions are required to construct q on the whole graph.

4. Proofs

4.1. Proof of Lemma 1. Consider the Sturm-Liouville equation (1) on the tree G . Let $C_j(x_j, \lambda)$ and $S_j(x_j, \lambda)$ be solutions of (1) on the edge e_j under initial conditions

$$C_j(0, \lambda) = S'_j(0, \lambda) = 1, \quad C'_j(0, \lambda) = S_j(0, \lambda) = 0.$$

Any solution $y = [y_j]_{j=1}^m$ of the equation (1) on G admits the following representation

$$y_j(x_j, \lambda) = M_j^0(\lambda)C_j(x_j, \lambda) + M_j^1(\lambda)S_j(x_j, \lambda), \quad j = \overline{1, m}, \quad x_j \in [0, T_j]. \quad (13)$$

Let BC be some fixed boundary conditions in the vertices $v \in \partial G$ of the form $y(v) = 0$ or $y'(v) = 0$ (for instance, we consider conditions (3) for the problem L and (4) for the problem L_k). Denote by L the boundary value problem for the Sturm-Liouville equation (1) with the standard matching conditions (2) and the boundary conditions BC . If y is a solution of a boundary value problem L , substitute (13) into (2) and BC , and obtain a linear algebraic system with respect to $M_j^0(\lambda)$, $M_j^1(\lambda)$. It is easy to check that the determinant of this system is a characteristic function $\Delta(\lambda)$ of the boundary value problem L , i.e. zeros of $\Delta(\lambda)$ coincide with the eigenvalues of L .

Example 1. Consider the problem L_0 for the star-type graph for $m = 3$. Then boundary conditions (3) yield $M_1^0(\lambda) = M_2^0(\lambda) = M_3^0(\lambda) = 0$. Consequently, from (2) we obtain the system with respect to $M_j^1(\lambda)$, $j = 1, 2, 3$, with the determinant

$$\Delta_0(\lambda) = \begin{vmatrix} S_1(T_1, \lambda) & -S_2(T_2, \lambda) & 0 \\ 0 & S_2(T_2, \lambda) & -S_3(T_3, \lambda) \\ S'_1(T_1, \lambda) & S'_2(T_2, \lambda) & S'_3(T_3, \lambda) \end{vmatrix}.$$

In the general case, the following assertion is valid.

Lemma 4. Let $w \in V$ and the degree of w be equal n . Splitting the vertex w , we split G into n subtrees G_i , $i = \overline{1, n}$. For each $i = \overline{1, n}$, let $\Delta_i^D(\lambda)$ and $\Delta_i^N(\lambda)$ be characteristic functions for boundary value problems for equation (1) on tree G_i with matching conditions (2), boundary conditions BC for $v \in \partial G \cap \partial G_i$ and the Dirichlet condition $y(u) = 0$ for $\Delta_i^D(\lambda)$ and the Neumann condition $y'(u) = 0$ for $\Delta_i^N(\lambda)$. Then the characteristic function $\Delta(\lambda)$ for G with the conditions (2) and BC admits the following representation:

$$\Delta(\lambda) = \begin{vmatrix} \Delta_1^D(\lambda) & -\Delta_2^D(\lambda) & 0 & \dots & 0 \\ 0 & \Delta_2^D(\lambda) & -\Delta_3^D(\lambda) & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -\Delta_n^D(\lambda) \\ \Delta_1^N(\lambda) & \Delta_2^N(\lambda) & \Delta_3^N(\lambda) & \dots & \Delta_n^N(\lambda) \end{vmatrix}. \quad (14)$$

Indeed, if we write the determinant for $\Delta(\lambda)$ and analyze the participation of the edges of G_i in this determinant, we can easily see that $\Delta(\lambda) = \Delta_i^D(\lambda)D_i(\lambda) + \Delta_i^N(\lambda)E_i(\lambda)$, where the functions $D_i(\lambda)$ and $E_i(\lambda)$ do not depend on the subtree G_i . Thus we can consider the simplest case of the star-type graph, when each G_i contains only one edge, and then change the multipliers, corresponding to subgraphs G_i , to $\Delta_i^D(\lambda)$ and $\Delta_i^N(\lambda)$. Thus we directly obtain (14) from the formula for the star-type graph.

Lemma 1 follows from Lemma 14 for the graph in the fig. 1. Alternatively, one can derive (6) from (21), changing characteristic functions for one-edge subtrees by general characteristic function.

4.2. Proof of Lemma 2. Together with L consider the boundary value problem L^0 for equation (1) with $q \equiv 0$, the matching conditions (2) and the boundary conditions BC . If some symbol γ denotes the object related to L , we denote by the symbol γ^0 the similar object related to L^0 . In particular, $\Delta^0(\lambda)$ is the characteristic function of L^0 . Let the symbol $P(\rho)$ stand for different polynomials of $\sin \rho T_j$ and $\cos \rho T_j$, $j = \overline{1, m}$.

Lemma 5. *The characteristic function $\Delta(\lambda)$ has the following asymptotic behavior:*

$$\Delta(\lambda) = \Delta^0(\lambda) + O(\rho^{-d}) = \frac{P(\rho)}{\rho^{d-1}} + O(\rho^{-d}), \quad \rho \in S_\delta, |\rho| \rightarrow \infty,$$

where $P(\rho) \not\equiv 0$ and $d = m - i - n$, where m is the number of the edges, i is the number of internal vertices and n is the number of boundary vertices with the Neumann boundary condition $y'(v) = 0$.

Proof. The claim of the lemma immediately follows from the standard asymptotic formulas

$$C_j(x_j, \lambda) = \cos \rho x_j + O(\rho^{-1}), \quad C'_j(x_j, \lambda) = -\rho \sin \rho x_j + O(1),$$

$$S_j(x_j, \lambda) = \frac{\sin \rho x_j}{\rho} + O(\rho^{-2}), \quad S'_j(x, \lambda) = \cos \rho x_j, \quad \rho \in S_\delta, |\rho| \rightarrow \infty,$$

and the construction of $\Delta(\lambda)$. The relation $P(\rho) \not\equiv 0$ follows from the regularity of the standard matching conditions. \square

Applying Lemma 5 to the characteristic functions, defined on Step 3 of the algorithm, we derive asymptotic representations for the coefficients $c = a_{ij}, b_{ij}, A, B, C$ in the following form:

$$c(\lambda) = c^0(\lambda) + O(\rho^{-d}) = \frac{P(\rho)}{\rho^{d-1}} + O(\rho^{-d}), \quad \rho \in S_\delta, |\rho| \rightarrow \infty,$$

where d stands for different integers. This relation yields Lemma 2.

4.3. Proof of Lemma 3. In this subsection, we consider only the problem L^0 with $q \equiv 0$, so we omit the index 0 for brevity. For simplicity, let $T_f = 1$. Taking into account, that

$$\Delta_3^{DD} = \frac{\sin \rho}{\rho}, \quad \Delta_3^{ND} = \Delta_3^{DN} = \cos \rho, \quad \Delta_3^{NN} = -\rho \sin \rho$$

and doing some algebra with the expressions (6), (8), (11), we derive

$$A(\lambda) = -F_1(\lambda)F_4(\lambda)\Delta_0(\lambda)\frac{\sin^2 \rho}{\rho^2}\Delta_4^{DD}(\lambda)\Delta_5^D(\lambda)\chi(\lambda), \quad (15)$$

$$B(\lambda) = -F_1(\lambda)F_4(\lambda)\frac{\sin \rho}{\rho}\Delta_0(\lambda)\left\{\Delta_5^D(\lambda)\Pi(\lambda) + \Delta_5^D(\lambda)\frac{\sin \rho}{\rho}\xi(\lambda) - \Delta_4^{DD}(\lambda)\Delta_5^N(\lambda)\frac{\sin \rho}{\rho}\chi(\lambda)\right\}, \quad (16)$$

$$C(\lambda) = F_1(\lambda)F_4(\lambda)\Delta_0(\lambda)\frac{\sin \rho}{\rho}\Delta_5^N(\lambda)\left\{\Pi(\lambda) + \frac{\sin \rho}{\rho}\xi(\lambda)\right\}, \quad (17)$$

where

$$F_i(\lambda) = \Delta_i^{DD}(\lambda)\Delta_i^{NN}(\lambda) - \Delta_i^{DN}(\lambda)\Delta_i^{ND}(\lambda), \quad i = 1, 4, \\ \Pi(\lambda) = 2\Delta_1^{DD}(\lambda)\Delta_2^{DD}(\lambda)\Delta_4^{DD}(\lambda),$$

$\chi(\lambda)$ and $\xi(\lambda)$ are characteristic functions of the graphs $G_1 \cup G_2 \cup G_3$ and $G_1 \cup G_2 \cup G_3 \cup G_4$, respectively. Here we mean that the copies of the vertex v_3 (and v_6 in the second graph) are joined into one vertex with the standard matching conditions (2).

Lemma 6. *Let v_1 and v_2 be two fixed vertices from ∂G . Denote by $\Delta^{DD}(\lambda)$, $\Delta^{DN}(\lambda)$, $\Delta^{ND}(\lambda)$ and $\Delta^{NN}(\lambda)$ the characteristic functions for equation (1) on the tree G with the matching conditions (2), with the following boundary conditions:*

$$\begin{aligned} \Delta^{DD}(\lambda): & \quad y(v_1) = y(v_2) = 0, \\ \Delta^{DN}(\lambda): & \quad y(v_1) = y'(v_2) = 0, \\ \Delta^{ND}(\lambda): & \quad y'(v_1) = y(v_2) = 0, \\ \Delta^{NN}(\lambda): & \quad y'(v_1) = y'(v_2) = 0, \end{aligned}$$

and with the conditions BC in the vertices $v \in \partial G \setminus \{v_1, v_2\}$. Then

$$\Delta^{DD}(\lambda)\Delta^{NN}(\lambda) - \Delta^{DN}(\lambda)\Delta^{ND}(\lambda) \neq 0. \quad (18)$$

Proof. We shall divide the proof into the following steps. 1. Let the tree G consists of the only edge $[v_1, v_2]$. Then one can check the relation (18) by direct calculation.

2. Let the vertices v_1 and v_2 be connected by edges with the same vertex v , and let there also be subtrees G_i , $i = \overline{1, n}$, from the vertex v (see fig. 2). Denote by $\Delta_i^D(\lambda)$ and $\Delta_i^N(\lambda)$ the characteristic functions for G_i with the matching conditions (2), the boundary conditions BC and $y(v) = 0$ for $\Delta_i^D(\lambda)$ and $y'(v) = 0$ for $\Delta_i^N(\lambda)$. According to Lemma 4, the following relation holds

$$\Delta^{DD}(\lambda) = \frac{\sin \rho T_1 \sin \rho T_2}{\rho^2} \Delta^K(\lambda) + \frac{1}{\rho} (\sin \rho T_1 \cos \rho T_2 + \cos \rho T_1 \sin \rho T_2) \Delta^\Pi(\lambda),$$

where

$$\Delta^\Pi(\lambda) = \prod_{i=1}^n \Delta_i^D(\lambda), \quad \Delta^K(\lambda) = \Delta^\Pi(\lambda) \sum_{i=1}^n \frac{\Delta_i^N(\lambda)}{\Delta_i^D(\lambda)}.$$

Using similar representations for $\Delta^{NN}(\lambda)$, $\Delta^{DN}(\lambda)$ and $\Delta^{ND}(\lambda)$, we derive

$$\Delta^{DD}(\lambda)\Delta^{NN}(\lambda) - \Delta^{DN}(\lambda)\Delta^{ND}(\lambda) = -(\Delta^\Pi(\lambda))^2 \neq 0.$$

3. Now let the vertices v_1 and v_2 be connected by the edges with v_3 and v_4 , respectively. Let the tree G splits by the vertices v_3 and v_4 into the subtrees G_i , $i = \overline{1, n_1}$, connected with v_3 , the subtrees \tilde{G}_j , $j = \overline{1, n_2}$, connected with v_4 , the subtree G_0 , including the both vertices v_3 and v_4 , and the edges e_1 , e_2 (see fig. 3). Denote by $\Delta_i^D(\lambda)$, $\Delta_i^N(\lambda)$, $i = \overline{1, n_1}$, and by $\tilde{\Delta}_j^D(\lambda)$, $\tilde{\Delta}_j^N(\lambda)$, $j = \overline{1, n_2}$, the characteristic functions for the subtrees G_i with the Dirichlet or Neumann boundary condition in v_3 and for the subtrees \tilde{G}_j with the Dirichlet or Neumann boundary condition in v_4 , respectively. Let $\Delta_0^{DD}(\lambda)$, $\Delta_0^{DN}(\lambda)$, $\Delta_0^{ND}(\lambda)$ and $\Delta_0^{NN}(\lambda)$ be characteristic functions for the subtree G_0 with the following boundary conditions

$$\begin{aligned} \Delta_0^{DD}(\lambda): & \quad y(v_3) = y(v_4) = 0, \\ \Delta_0^{DN}(\lambda): & \quad y(v_3) = y'(v_4) = 0, \\ \Delta_0^{ND}(\lambda): & \quad y'(v_3) = y(v_4) = 0, \\ \Delta_0^{NN}(\lambda): & \quad y'(v_3) = y'(v_4) = 0, \end{aligned}$$

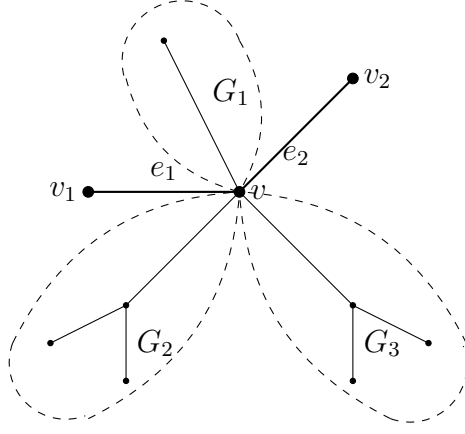


Figure 2

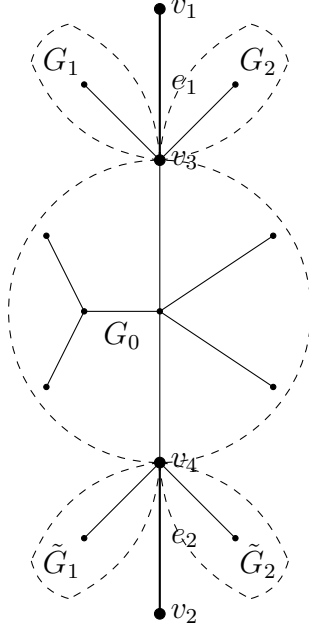


Figure 3

and the conditions BC in other boundary vertices. Denote the functions

$$\begin{aligned}
\Delta_1^\Pi(\lambda) &= \prod_{i=1}^{n_1} \Delta_i^D(\lambda), \quad \Delta_2^\Pi(\lambda) = \prod_{j=1}^{n_2} \tilde{\Delta}_j^D(\lambda), \\
\Delta_1^K(\lambda) &= \Delta_1^\Pi(\lambda) \sum_{i=1}^n \frac{\Delta_i^N(\lambda)}{\Delta_i^D(\lambda)}, \quad \Delta_2^K(\lambda) = \Delta_2^\Pi(\lambda) \sum_{j=1}^n \frac{\tilde{\Delta}_j^N(\lambda)}{\tilde{\Delta}_j^D(\lambda)}. \\
\left. \begin{aligned}
\Delta^{KK}(\lambda) &= \Delta_0^{DD}(\lambda) \Delta_1^K(\lambda) \Delta_2^K(\lambda) + \Delta_0^{ND}(\lambda) \Delta_1^\Pi(\lambda) \Delta_2^K(\lambda) \\
&\quad + \Delta_0^{DN}(\lambda) \Delta_1^K(\lambda) \Delta_2^\Pi(\lambda) + \Delta_0^{NN}(\lambda) \Delta_1^\Pi(\lambda) \Delta_2^\Pi(\lambda), \\
\Delta^{\Pi K}(\lambda) &= \Delta_0^{DD}(\lambda) \Delta_1^\Pi(\lambda) \Delta_2^K(\lambda) + \Delta_0^{DN}(\lambda) \Delta_1^\Pi(\lambda) \Delta_2^\Pi(\lambda), \\
\Delta^{K\Pi}(\lambda) &= \Delta_0^{DD}(\lambda) \Delta_1^K(\lambda) \Delta_2^\Pi(\lambda) + \Delta_0^{ND}(\lambda) \Delta_1^K(\lambda) \Delta_2^\Pi(\lambda), \\
\Delta^{\Pi\Pi}(\lambda) &= \Delta_0^{DD}(\lambda) \Delta_1^\Pi(\lambda) \Delta_2^\Pi(\lambda).
\end{aligned} \right\} \quad (19)
\end{aligned}$$

In view of Lemma 4, the following relation holds

$$\Delta^{DD}(\lambda) = \frac{\sin \rho T_1 \sin \rho T_2}{\rho^2} \Delta^{KK}(\lambda) + \frac{\cos \rho T_1 \sin \rho T_2}{\rho} \Delta^{\Pi K}(\lambda)$$

$$+ \frac{\sin \rho T_1 \cos \rho T_2}{\rho} \Delta^{K\Pi}(\lambda) + \cos \rho T_1 \cos \rho T_2 \Delta^{\Pi\Pi}(\lambda).$$

Together with the similar relations for $\Delta^{DN}(\lambda)$, $\Delta^{ND}(\lambda)$ and $\Delta^{NN}(\lambda)$, it yields

$$\Delta^{DD}(\lambda)\Delta^{NN}(\lambda) - \Delta^{DN}(\lambda)\Delta^{ND}(\lambda) = \Delta^{\Pi\Pi}(\lambda)\Delta^{KK}(\lambda) - \Delta^{\Pi K}(\lambda)\Delta^{K\Pi}(\lambda)$$

Taking (19) into account, we obtain

$$\Delta^{\Pi\Pi}(\lambda)\Delta^{KK}(\lambda) - \Delta^{\Pi K}(\lambda)\Delta^{K\Pi}(\lambda) = (\Delta_0^{DD}(\lambda)\Delta_0^{NN}(\lambda) - \Delta_0^{DN}(\lambda)\Delta_0^{ND}(\lambda)) (\Delta_1^{\Pi}(\lambda)\Delta_2^{\Pi}(\lambda))^2.$$

By virtue of Lemma 5, $\Delta_i^{\Pi}(\lambda) \not\equiv 0$, $i = 1, 2$. Therefore the relation (18) holds for the tree G if and only if it holds for the subtree G_0 . By induction, the claim of the lemma is valid for any tree G . \square

By virtue of Lemmas 5, 6 and (15), $A(\lambda) \not\equiv 0$. It follows from (15), (16), (17), that

$$\begin{aligned} D(\lambda) &= B^2(\lambda) - 4A(\lambda)C(\lambda) \\ &= F_1^2(\lambda)F_4^2(\lambda)\frac{\sin^2 \rho}{\rho^2}\Delta_0(\lambda) \left\{ \Delta_5^D(\lambda)\Pi(\lambda) + \Delta_5^D(\lambda)\frac{\sin \rho}{\rho}\xi(\lambda) + \Delta_4^{DD}(\lambda)\Delta_5^N(\lambda)\frac{\sin \rho}{\rho}\chi(\lambda) \right\}^2. \end{aligned}$$

Note that the expression in the bracket above equals to

$$\Delta_5^D(\lambda)\Pi(\lambda) + \frac{\sin \rho}{\rho}\Delta_0(\lambda).$$

Similarly to Lemma 5, the following asymptotic formulas can be obtained:

$$\Delta_5^D(\lambda)\Pi(\lambda) = C_1 r^{-p} \exp(r(T-1))[1], \quad \frac{\sin \rho}{\rho}\Delta_0(\lambda) = C_2 r^{-q} \exp(r(T+1))[1],$$

where $\rho = ir$, $r \rightarrow +\infty$, $T = \sum_{j=1}^m T_j$, C_1 , C_2 , p and q are some constants. Clearly, the second term grows faster than the first one. Therefore $\Delta_0(\lambda) \not\equiv 0$ implies $D(\lambda) \not\equiv 0$. The proof of Lemma 3 is finished.

Using Lemma 5, one can also check, that $B(\lambda)$ and $\sqrt{D(\lambda)}$ have the same power of ρ in the denominator, so the roots of (10) have different asymptotic behavior.

5. Example

In this section, we provide the solution of Inverse Problem 4 for the example of the graph in the fig. 4. For simplicity, let $T_j = 1$, $j = \overline{1, 5}$. Let $x_3 = 0$ corresponds to the vertex v_3 and $x_3 = 1$ corresponds to v_6 . For the boundary edges, $x_j = 0$ correspond to the boundary vertices. The matching conditions (2) take the form

$$\begin{aligned} v_3: \quad & y_1(1) = y_2(1) = y_3(0), \quad y'_1(1) + y'_2(1) - y'_3(0) = 0, \\ v_6: \quad & y_3(1) = y_4(1) = y_5(1), \quad y'_3(1) + y'_4(1) + y'_5(1) = 0. \end{aligned} \tag{20}$$

For this example, each subtree G_i consists of only one edge e_i , $i = \overline{1, 5}$. Let us know the spectra Λ_0 , Λ_1 , Λ_4 and the potential q_3 . Using the given spectra, one can easily find the characteristic functions $\Delta_0(\lambda)$, $\Delta_1(\lambda)$, $\Delta_4(\lambda)$ and the Weyl functions $M_1(\lambda)$, $M_4(\lambda)$. Solving problems IP(1) and IP(4), recover q_1 and q_3 .

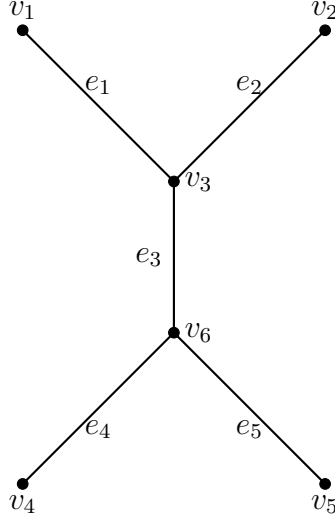


Figure 4. Example

Consider the boundary value problem L . Represent the solution y in the form (13) and substitute it into (2) and (3). From (3), one gets $M_1^0(\lambda) = M_2^0(\lambda) = M_4^0(\lambda) = M_5^0(\lambda) = 0$. Then matching conditions (20) yield the system

$$\begin{pmatrix} S_1 & -S_2 & 0 & 0 & 0 & 0 \\ 0 & S_2 & -1 & 0 & 0 & 0 \\ S'_1 & S'_2 & 0 & -1 & 0 & 0 \\ 0 & 0 & C_3 & S_3 & -S_4 & 0 \\ 0 & 0 & 0 & 0 & S_4 & -S_5 \\ 0 & 0 & C'_3 & S'_3 & S'_4 & S'_5 \end{pmatrix} \begin{pmatrix} M_1^1 \\ M_2^1 \\ M_3^0 \\ M_3^1 \\ M_4^1 \\ M_5^1 \end{pmatrix} = 0. \quad (21)$$

Here we omit arguments $(1, \lambda)$ and (λ) for brevity. The characteristic function $\Delta_0(\lambda)$ equals the determinant of (21). Since we know q_1 , q_3 and q_4 , we can solve (1) and obtain the functions $S_j(x_j, \lambda)$ and $C_j(x_j, \lambda)$ for $j = 1, 3, 4$. Therefore the determinant admits the following representation

$$\Delta_0 = a_{11}S_2S_5 + a_{12}S'_2S_5 + a_{13}S_2S'_5 + a_{14}S'_2S'_5,$$

where

$$\begin{aligned} a_{11} &= S'_1 \begin{vmatrix} S_3 & -S_4 \\ S'_3 & S'_4 \end{vmatrix} + S_1 \begin{vmatrix} C_3 & -S_4 \\ C'_3 & S'_4 \end{vmatrix}, & a_{12} &= S_1 \begin{vmatrix} S_3 & -S_4 \\ S'_3 & S'_4 \end{vmatrix}, \\ a_{13} &= (S'_1S_3 + S_1C_3)S_4, & a_{14} &= S_1S_3S_4. \end{aligned}$$

If one change S_1 to C_1 or S_4 to C_4 , he obtains analogous relations for $\Delta_1(\lambda)$ and $\Delta_4(\lambda)$, respectively. Thus we arrive at the system (7).

Let $q \equiv 0$ on G . Then

$$\begin{aligned} C_j^0(x_j, \lambda) &= \cos \rho x_j, & S_j^0(x_j, \lambda) &= \frac{\sin \rho x_j}{\rho}, \\ a_{11}^0 &= \frac{\sin 3\rho}{\rho}, & a_{12}^0 &= a_{13}^0 = \frac{\sin 2\rho \sin \rho}{\rho^2}, & a_{14}^0 &= \frac{\sin^3 \rho}{\rho^3}, \\ a_{21}^0 &= a_{31}^0 = \cos 3\rho, & a_{22}^0 &= a_{33}^0 = \frac{\sin 2\rho \cos \rho}{\rho}, \\ a_{23}^0 &= a_{32}^0 = \frac{\cos 2\rho \sin \rho}{\rho}, & a_{24}^0 &= a_{34}^0 = \frac{\cos \rho \sin^2 \rho}{\rho^2}. \end{aligned}$$

$$\Delta_0^0 = \frac{-9 \sin 5\rho + 13 \sin 3\rho + 6 \sin \rho}{16\rho^3}, \quad \Delta_1^0 = \Delta_4^0 = \frac{-9 \cos 5\rho + 7 \cos 3\rho + 2 \cos \rho}{16\rho^2}.$$

Using (8), we obtain

$$b_{11}^0 = b_{21}^0 = \frac{-3 \sin 6\rho - 2 \sin 4\rho + 13 \sin 2\rho}{16\rho^3}, \quad b_{12}^0 = b_{23}^0 = \frac{-3 \cos 6\rho + 6 \cos 4\rho + 3 \cos 2\rho - 6}{16\rho^4},$$

$$b_{13}^0 = b_{22}^0 = \frac{3 \cos 6\rho - 10 \cos 4\rho + 13 \cos 2\rho - 6}{32\rho^4}, \quad b_{14}^0 = b_{24}^0 = \frac{-3 \sin 6\rho + 12 \sin 4\rho - 15 \sin 2\rho}{32\rho^5}.$$

Substitute these formulas into (11):

$$A_0 = \frac{-27 \sin 12\rho + 174 \sin 10\rho - 420 \sin 8\rho + 378 \sin 6\rho + 153 \sin 4\rho - 468 \sin 2\rho}{2048\rho^9},$$

$$B_0 = \frac{-27 \cos 12\rho + 84 \cos 10\rho + 106 \cos 8\rho - 764 \cos 6\rho + 1099 \cos 4\rho - 344 \cos 2\rho - 154}{2048\rho^8},$$

$$C_0 = \frac{-27 \sin 12\rho + 48 \sin 10\rho + 140 \sin 8\rho - 336 \sin 6\rho - 71 \sin 4\rho + 512 \sin 2\rho}{1024\rho^7}.$$

Calculate the discriminant of equation (12):

$$D_0 = B_0^2 - 4A_0C_0 = (6561 \cos 24\rho - 52488 \cos 22\rho + 128628 \cos 20\rho + 83592 \cos 18\rho \\ - 987134 \cos 16\rho + 1543976 \cos 14\rho + 702372 \cos 12\rho - 4646312 \cos 10\rho + 3755087 \cos 8\rho \\ + 3053616 \cos 6\rho - 4805144 \cos 4\rho - 4176688 \cos 2\rho + 5393934)/(8388608\rho^{16}).$$

We used wxMaxima 12.04.0 for calculations.

Obviously, $A_0(\lambda) \neq 0$, $D_0(\lambda) \neq 0$, so according to Lemma 2, the roots of equation (10) in the general case have different asymptotics:

$$\tilde{M}_2^1(\lambda) = \frac{\rho \cos \rho}{\sin \rho}[1], \quad \tilde{M}_2^2(\lambda) = -\frac{1 + 6 \cos^2 \rho}{3 \sin \rho \cos \rho}[1].$$

Since $\tilde{M}_2(\lambda) = \frac{S_2'(1, \lambda)}{S_2(1, \lambda)}$, only the root $\tilde{M}_2^1(\lambda)$ is the required one.

Finally, one can easily find $\tilde{M}_5(\lambda)$ and solve classical Sturm-Liouville inverse problems by Weyl functions on the edges e_2 and e_5 .

Now let us consider the case when the potential is known a priori on two edges. If they are e_1 and e_4 , then only two spectra Λ_0 and Λ_2 are sufficient to recover the potential on the whole graph. Indeed, one can solve IP(2), then apply Theorem 1 to the vertex v_3 , find q_3 and then similarly find q_5 . However, the knowledge of q_1 and q_2 do not allow us to recover the potential from two spectra by our method. If we have only Λ_0 and Λ_4 , we can not recover q_3 . Similarly, if we know q_3 initially, the knowledge of the potential on one of the boundary edges do not allow us to reduce the number of given spectra. Thus, if the potential is known on multiple edges, the number of required spectra depends on the location of these edges.

Acknowledgments. This work was supported by Grant 1.1436.2014K of the Russian Ministry of Education and Science, by Grants 15-01-04864 and 16-01-00015 of Russian Foundation for Basic Research and by the Mathematics Research Promotion Center of Taiwan.

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